S.-T. Yau College Student Mathematics Contests 2017

Oral Exam of Geometry and Topology

Individual Problems

1. Show that $S^2 \times S^2$ and $\mathbb{CP}^2 \vee S^2$ are not homotopically equivalent.

2 (Bonnet-Myers theorem)**.** Prove that a complete Riemannian manifold M whose sectional curvature is everywhere bounded below by a constant k has diameter at most $\pi/\sqrt{k}$. In particular, M is compact.

3 (Isoperimetric inequality)**.** For the length L of a closed curve and the area A of the planar region that it encloses, show that $4\pi A \leq L^2$ and that equality holds if and only if the curve is a circle.

4. (a) Compute the cohomology of the unitary group $U(n)$.
(b) Compute $\pi_1(U(n))$ and $\pi_2(U(n))$.

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Team Problems

1. (a) Describe the loop space ΩS^2 and path space PS^2 of the sphere S^2 in the following fibration:

$$\begin{array}{ccc} \Omega S^2 & \longrightarrow & PS^2 \\ & & \downarrow \\ & & S^2. \end{array}$$

(b) Compute the cohomology of the loop space ΩS^2. What is the ring structure of $H^*(\Omega S^2)$?

2 (Synge theorem)**.** Let M be an even-dimensional compact Riemannian manifold with positive sectional curvature.

(a) When M is orientable, show that M is simply connected.

(b) When M is unorientable, what is $\pi_1(M)$?

3. (a) Let C be a smooth curve on the sphere. The Crofton formula expresses the arc length $L(C)$ of the curve C as

$$L(C) = \frac{1}{4}\int_{S^2} n(C \cap W^\perp)dW.$$

Here $W^\perp$ is the plane with normal W going through the origin and $n(C \cap W^\perp)$ is the number of points in the intersection of C and $W^\perp$.

(b) Sketch a proof of Crofton formula.

4. Let $\Omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ and $\alpha > 0$. Suppose D is the surface in $\mathbb{R}^3$ defined by $\{(x,y,z) \,|\, x^2+y^2+z^2 = 1, z \geq \alpha\sqrt{x^2+y^2}\}$.

(a) Show that $\Omega|_D$ is an orientation form and makes D an oriented manifold with boundary.

(b) Evaluate $\int_D \Omega$. Your answer should be in terms of α.

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Overall Problems

1. Let Σ_g be a compact Riemann surface of genus $g > 1$, $Aut(\Sigma_g)$ be the automorphism group of biholomorphic maps of Σ_g. Let $V = H^0(\Sigma_g, K)$ be the space of holomorphic 1-forms on Σ_g.

(a) Show that the natural group homomorphism

$$\rho : Aut(\Sigma_g) \to GL(V)$$

is injective.

(b) V carries a natural hermitian structure

$$< \omega_1, \omega_2 >= i \int_{\Sigma_g} \omega_1 \wedge \overline{\omega_2}, \quad \omega_i \in V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside the unitary subgroup.

(c) V carries a natural integral structure from the lattice

$$H^1(\Sigma, \mathbb{Z})(\simeq \mathbb{Z}^{2g}) \subset V.$$

Show that $\rho(Aut(\Sigma_g))$ lies inside $GL(\mathbb{Z}^{2g})$.

(d) Conclude that $Aut(\Sigma_g)$ is a finite group.

2. (a) What is a Killing field on a Riemannian manifold?

(b) Explain why a Killing field on a connected Riemannian manifold is determined by its value and the value of its first derivative at a given point.

(c) Show that the maximal dimension of the space of Killing fields on a three dimensional connected Riemannian manifold is six.

3. (a) Let X be an n-dimensional compact Riemannian manifold. Show that

$$\dim(\mathrm{Isom}(X)) \leq \frac{n(n+1)}{2}.$$

(b) List all possible M when the equality in the above is achieved.